A METHOD OF SOLVING LINEAR DIFFERENTIAL EQUATIONS *

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The object of this paper is to present a new method of solving ordinary linear diffential equations, which may frequently be applied with advantage when the coefficients of the equation are polynomials in the independent variable.

Let the given differential equation be

$$f\left(x,\, y,\, \frac{dy}{dx}\,,\, \frac{d^2y}{dx^2},\, \cdots\, \frac{d^ny}{dx^n}\right) = 0. \tag{1}$$

The method of solution proposed consists of the following steps:

(a) Break up the function f into two parts, one of which, f_1, equated to zero gives a differential equation which may be readily solved, and introduce a parameter t as a factor of the second part so that the given equation, $f_1 + f_2 = 0$, is replaced by

$$f_1 + tf_2 = 0. \tag{2}$$

(b) Assume that the series

$$y = A + Bt + Ct^2 + Dt^3 \cdots, \tag{3}$$

where A, B, C, D, $\cdots$ are undetermined functions of x, satisfies (2). Substitute the expression (3) for y in (2) and determine these functions by solving the differential equations formed by equating to zero the coefficients of successive powers of t in this identity.

(c) Substitute the values of A, B, C, D, $\cdots$ in (3), and replace t by unity, and see whether

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$$y = A + B + C + D \cdots \tag{4}$$

satisfies equation (1).

It is to be noted that A is a solution of the equation $f_1 = 0$, and may consequently be frequently determined by a suitable choice of f_1 to contain n arbitrary constants so leading to the general solution of (1). If this is possible it is necessary only to obtain particular solutions of the equations for B, C, D, $\cdots$.

Since the process above described is purely formal it is evidently necessary to see whether or not the series (4) actually satisfies (1) if that series contains an infinite number of terms, or if any of the functions A, B, C, $\cdots$ is given by an infinite series.

The most advantageous method of breaking up the given equation into two parts must be determined by trial. However, if no term is separated into two parts, the number of possible methods of choosing f_1 and f_2 is never greater than 2^n, and if n is not large the best method may be selected without much difficulty.

In the process of solving the differential equations which determine A, B, C, D, $\cdots$, independent arbitrary constants are introduced until the number of arbitrary constants equals the order of the given differential equation.

If the independent arbitrary constants are C_1, C_2, C_3, $\cdots$ C_n the terms of the series (4) may be grouped so that (4) takes the form

$$y = C_1 y_1 + C_2 y_2 + C_3 y_3 + \cdots + C_n y_n + Y, \tag{5}$$

where y_1, y_2, y_3, $\cdots$ y_n are independent solutions of the corresponding homogeneous differential equation, and Y is a particular integral of (1).

If another manner of breaking up the given differential equation makes series (4) a solution of (1), there may be determined a different set of independent integrals and a different particular integral.

If the general solution (5) of the differential equation contains infinite series, the limits of the regions of convergency must be determined by the usual methods.

This method of solving differential equations is the result of an attempt to extend to differential equations the method employed by the author in the papers "On the solution of algebraic equations in infinite series." †

† *Bulletin American Mathematical Society*, ser. 2, vol. 14, 1908, pp. 467–477. *Proceedings American Philosophical Society*, vol. 47, 1908, pp. 111–134.

The incentive to make this attempt came from the following statement in an extract of a letter from Cauchy to Coriolis of January 29, 1837, published in the *Comptes Rendus* of the Paris Academy.

"Ainsi étendus, ces mèthodes s'appliquent avec un succès remarquable à presque tous les grands problèmes d'analyse, à la résolution générale des équations, à l'intégration des équations différentielles, à la mécanique céleste, etc."

Cauchy describes the method applied to algebraic equations as follows:

"Pour résoudre une équation partagez son premier membre en deux polynomes d'une manière quelconque, et supposez l'un de ces polynomes multiplié par un paramètre que vous reduisez plus tard à l'unité."

By this method the solutions of Bessel's equations, of Legendre's equation, and of the differential equation of the hypergeometric series may be advantageously determined.

The method will be examplified by applying it to two differential equations.

Example 1. Solve

$$\frac{d^2y}{dx^2} + ax^2y = 1 + x.$$

Writing this equation in the form

$$\frac{d^2y}{dx^2} - (1 + x) + ax^2yt = 0,$$

and assuming that

$$y = A + Bt + Ct^2 + Dt^3 + \cdots$$

there results

$$\left.\begin{array}{r}\dfrac{d^2A}{dx^2}\\ -(1+x)\end{array}\right| + \left.\begin{array}{r}\dfrac{d^2B}{dx^2}\\ +ax^2A\end{array}\right| t + \left.\begin{array}{r}\dfrac{d^2C}{dx^2}\\ +ax^2B\end{array}\right| t^2 + \left.\begin{array}{r}\dfrac{d^2D}{dx^2}\\ +ax^2C\end{array}\right| t^3 + \cdots \equiv 0.$$

Equating to zero the coefficients of the successive powers of t in this identity, we have

$$\frac{d^2A}{dx^2} = 1 + x,$$

$$\frac{d^2B}{dx^2} + ax^2A = 0,$$

$$\frac{d^2C}{dx^2} + ax^2B = 0,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

From this series of differential equations

$$A = K_1 + K_2x + \frac{x^2}{1\cdot 2} + \frac{x^3}{2\cdot 3},$$

$$B = -K_1\frac{ax^4}{3\cdot 4} - K_2\frac{ax^5}{4\cdot 5} - \frac{ax^6}{1\cdot 2\cdot 5\cdot 6} - \frac{ax^7}{2\cdot 3\cdot 6\cdot 7},$$

$$C = K_1\frac{a^2x^8}{3\cdot 4\cdot 7\cdot 8} + K_2\frac{a^2x^9}{4\cdot 5\cdot 8\cdot 9} + \frac{a^2x^{10}}{1\cdot 2\cdot 5\cdot 6\cdot 9\cdot 10} + \frac{a^2x^{11}}{2\cdot 3\cdot 6\cdot 7\cdot 10\cdot 11},$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

The law of formation of the successive coefficients A, B, C, D, $\cdots$ is evident, and the value of y, when t is made unity, becomes

$$\begin{aligned} y = K_1 & \left(1 - \frac{ax^4}{3\cdot 4} + \frac{a^2x^8}{3\cdot 4\cdot 7\cdot 8} - \frac{a^3x^{12}}{3\cdot 4\cdot 7\cdot 8\cdot 11\cdot 12} + \cdots\right) \\ + K_2x & \left(1 - \frac{ax^4}{4\cdot 5} + \frac{a^2x^8}{4\cdot 5\cdot 8\cdot 9} - \frac{a^3x^{12}}{4\cdot 5\cdot 8\cdot 9\cdot 12\cdot 13} + \cdots\right) \\ + \frac{x^2}{2} & \left(1 - \frac{ax^4}{5\cdot 6} + \frac{a^2x^8}{5\cdot 6\cdot 9\ 10} - \cdots\right) \\ + \frac{x^3}{6} & \left(1 - \frac{ax^4}{6\cdot 7} + \frac{a^2x^8}{6\cdot 7\cdot 10\cdot 11} - \cdots\right). \end{aligned}$$

This value of y is composed of four infinite series, each convergent for all values of x, and is the complete solution of the given differential equation.

Example 2. Solve

$$x^2\frac{d^2y}{dx^2} + (x + 2x^2)\frac{dy}{dx} - 4y = 0 \cdot$$

Writing this equation in the form

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} - 4y + 2x^2 t \frac{dy}{dx} = 0,$$

and assuming that

$$y = A + Bt + Ct^2 + Dt^3 + Et^4 + \cdots,$$

there results

$$\begin{array}{r|r|r|l}
x^2 \dfrac{d^2A}{dx^2} & + x^2 \dfrac{d^2B}{dx^2} & t + x^2 \dfrac{d^2C}{dx^2} & t^2 + \cdots \equiv 0. \\
+ x \dfrac{dA}{dx} & + x \dfrac{dB}{dx} & + x \dfrac{dC}{dx} & \\
- 4A & - 4B & - 4C & \\
 & + 2x^2 \dfrac{dA}{dx} & + 2x^2 \dfrac{dB}{dx} &
\end{array}$$

Equating to zero the coefficients of the successive powers of t in this identity, we have

$$x^2 \frac{d^2A}{dx^2} + x \frac{dA}{dx} - 4A = 0,$$

$$x^2 \frac{d^2B}{dx^2} + x \frac{dB}{dx} - 4B + 2x \frac{dA}{dx} = 0,$$

$$x^2 \frac{d^2C}{dx^2} + x \frac{dC}{dx} - 4C + 2x^2 \frac{dB}{dx} = 0,$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

The solutions of the first equation for A are $A = K_1x^2$ and $A = K_2x^{-2}$.

(a) Substituting $A = K_1x^2$ in the next equation,

$$x^2 \frac{d^2B}{dx^2} + x \frac{dB}{dx} - 4B + 4K_1 x^3 = 0.$$

A particular solution of this differential equation must be determined. The particular solution may be found by multiplying this equation by x and by two successive direct integrations.

The particular solution may also be found by the method of this paper as follows: Writing the equation in the form

$$x^2\frac{d^2B}{dx^2} + x\frac{dB}{dx} + 4K_1x^3 - 4Bt = 0$$

and assuming that

$$B = B_1 + B_2t + B_3t^2 + B_4t^3 + \cdots,$$

there results

$$\begin{array}{r|r|r|l} x^2\dfrac{d^2B_1}{dx^2} & + x^2\dfrac{d^2B_2}{dx^2} & t + x^2\dfrac{d^2B_3}{dx^2} & t^2 + \cdots \equiv 0. \\ + x\dfrac{dB_1}{dx} & + x\dfrac{dB_2}{dx} & + x\dfrac{dB_3}{dx} & \\ + 4K_1x^3 & - 4B_1 & - 4B_2 & \end{array}$$

Equating to zero the coefficients of the successive powers of t in this identity, we have

$$x^2\frac{d^2B_1}{dx^2} + x\frac{dB_1}{dx} + 4K_1x^3 = 0,$$

$$x^2\frac{d^2B_2}{dx^2} + x\frac{dB_2}{dx} - 4B_1 = 0,$$

$$x^2\frac{d^2B_3}{dx^2} + x\frac{dB_3}{dx} - 4B_2 = 0,$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots$$

From this series of differential equations, we find the particular solutions

$$B_1 = -\frac{4}{9}K_1x^3,\ B_2 = -\left(\frac{4}{9}\right)^2K_1x^3,\ B_3 = -\left(\frac{4}{9}\right)^3x^3,\ \cdots.$$

Hence

$$B = -\frac{4}{9}K_1x^3 - \left(\frac{4}{9}\right)^2K_1x^3 - \left(\frac{4}{9}\right)^3K_1x^3 - \cdots = -\frac{4}{5}K_1x^3.$$

Substituting this value of B,

$$x^2\frac{d^2C}{dx^2} + x\frac{dC}{dx} - 4C - \frac{2\cdot3\cdot4}{5}K_1x^4 = 0.$$

A particular solution of this equation is

$$C = \frac{3 \cdot 4}{5 \cdot 6} K_1 x^4.$$

In like manner

$$D = -\frac{4 \cdot 8}{5 \cdot 6 \cdot 7} K_1 x^5, \quad E = \frac{5 \cdot 16}{5 \cdot 6 \cdot 7 \cdot 8} K_1 x^6, \cdots$$

$$N = (-1)^n \frac{(n+1)\, 2^n}{5 \cdot 6 \cdot 7 \cdot 8 \cdots (n+4)} x^{n+2}.$$

Substituting the values of A, B, C, D, $\cdots$, and making t unity,

$$y = K_1 x^2 \Big[1 - \frac{4}{5} x + \frac{3 \cdot 4}{5 \cdot 6} x^2 + \frac{4 \cdot 8}{5 \cdot 6 \cdot 7} x^3 + \cdots$$

$$+ (-1)^n \frac{(n+1)\, 2^n}{5 \cdot 6 \cdot 7 \cdot 8 \cdots (n+4)} x^n \cdots,$$

which is convergent for all values of x and a solution of the given differential equation.

(b) Substituting $A = K_2 x^{-2}$,

we have

$$x^2 \frac{d^2 B}{dx^2} + x \frac{dB}{dx} - 4B - 4K_2 x^{-1} = 0.$$

A particular solution of this equation, found by the same methods, is

$$B = -\frac{4}{3} K_2 x^{-1}.$$

Substituting this value of B

$$x^2 \frac{d^2 C}{dx^2} + x \frac{dC}{dx} - 4C + \frac{8}{3} K_2 = 0.$$

A particular solution of this equation is

$$C = \frac{2}{3} K_2.$$

It is evident that $D = 0$, $E = 0$ $\cdots$ are particular solutions of the remaining differential equations of the series.

Hence

$$y = K_2 \left(x^{-2} - \frac{4}{3} x^{-1} + \frac{2}{3} \right)$$

is a solution of the given differential equation.

The general solution of the given differential equation is the sum of these two independent integrals.

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